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ANALYSIS AND SYNTHESIS OF UNEQUALLY SPACED ARRAYS USING THE Z-TRANSFORM

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ABSTRACT

It is shown that under certain conditions, Z-transform theory can be applied to simplify analysis and synthesis of unequally spaced antenna arrays. The approach provides significant advantages over other techniques since the method yields closed-form expressions for array factors that are otherwise often cumbersome and difficult to work with analytically. The method is useful for both broadside and endfire arrays and allows many of the results originally derived for equally spaced arrays to be applied.

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I. INTRODUCTION

It has been shown that Z-transform theory can be applied to simplify analysis and synthesis of equally spaced antenna arrays.[1,2] The approach provides significant advantages over earlier techniques since the method yields closed-form expressions for array factors corresponding to many equally spaced and unequally excited arrays that are otherwise cumbersome and difficult to work with analytically. The purpose of this report is to show that under certain conditions Z-transform techniques can also be applied for the treatment of unequally spaced arrays.

II. THEORY

Consider a linear array of n isotropic point sources that are equally excited in amplitude and progressively phased. If the positions of the array elements are defined by a spacing function $f(s)$ the array factor can be written as

$$E = \sum_{k=0}^{n-1} e^{j\, f(k)\psi} \tag{1}$$

where

$$\psi = \beta d[\cos\theta - \cos\theta_o]$$

$$\beta = 2\pi/\lambda$$

d = nominal interelement spacing

θ = angle measured from the array axis

θ_o = angle designating direction of maximum radiation

λ = operating wavelength.

If the interelement spacings are nearly uniform across the array it is convenient to write

$$f(s) = s + A\, g(s) \tag{2}$$

$$|g(s)| \leq 1$$

so that the array factor becomes

$$E = \sum_{k=0}^{n-1} e^{jk\psi}\, e^{j\, A\, g(k)\, \psi} \tag{3}$$

If all deviations from uniform spacing are small then A is small and the small-angle approximations can be used to simplify (3). Thus,

$$E \approx \sum_{k=0}^{n-1} e^{jk\psi}\, [1 + jA\psi g(k)] \tag{4}$$

Rewriting (4) using the relationship

$$\psi = \frac{4}{\pi}\, [\sin \psi - \frac{\sin 3\psi}{(3)^2} + \frac{\sin 5\psi}{(5)^2} - \ldots]$$

$$= \frac{2}{j\pi}\, [(e^{j\psi} - e^{-j\psi}) - \frac{(e^{j3\psi} - e^{-j3\psi})}{9} + \ldots] \tag{5}$$

valid over $-\pi/2 < \psi < +\pi/2$, and substituting $z = e^{-j\psi}$

$$E \approx \sum_{k=0}^{n-1} z^{-k} - \frac{2A}{\pi}\, [(z-z^{-1}) - \frac{(z^3 - z^{-3})}{9} + \ldots] \sum_{k=0}^{n-1} g(k)\, z^{-k} \tag{6}$$

Recalling that the Z-transform of a function f(s) can be defined as

$$Z[f(s)] = \sum_{k=0}^{\infty} f(k)\, z^{-k} \qquad (7)$$

(6) can be rewritten in final form as

$$E \approx Z[u(s) - u(s-n)] - \frac{2A}{\pi}\left[(z-z^{-1}) - \frac{(z^{3}-z^{-3})}{9} + \ldots\right] Z[g(s)(u(s)-u(s-n))] \qquad (8)$$

where u(s) is the unit step function. When the function g(s)[u(s)-u(s-n)] is Z-transformable and when only a few of the terms in (5) are used, (8) will be in closed form. The final expression can then be analyzed using standard techniques.[1,2]

As an example consider the function

$$g(s) = \sin\left(\frac{2\pi s}{n-1}\right) \qquad (9)$$

defining a form of space-tapered array. If only the first term of (5) is used and if A is small to provide nearly uniform spacing it can be shown using (8) that

$$E \approx \frac{1-z^{-n}}{1-z^{-1}} - \frac{2A}{\pi}\, \frac{\sin\left(\frac{2\pi}{n-1}\right)\left[1-z^{-2} - z^{-(n-1)} + z^{-(n+1)}\right]}{\left(1 - 2\cos\left(\frac{2\pi}{n-1}\right) z^{-1} + z^{-2}\right)} \qquad (10)$$

an expression which is valid regardless of the total number of array elements.

In Z-transform notation the general problem of synthesizing an equally excited unequally spaced array to provide an array factor approximating that of a given unequally excited but uniformly spaced array is quite straight-forward. Suppose an equally spaced array with excitation amplitudes defined by the function h(s) is given. The array factor E_1 is given by

$$E_1 = Z[h(s)\,(u(s) - u(s-n))] \tag{11}$$

If the deviations from unity of the function h(s) are not too great an appropriately normalized equally excited unequally spaced array can be derived that provides an array factor approximating that given in (11). Equating (11) with (8) and solving for the spacing function yields

$$Z[Ag(s)\,(u(s) - u(s-n))] = \frac{\pi}{2}\,\frac{Z[(h(s)-1)(u(s) - u(s-n))]}{(z^{-1} - z)} \tag{12}$$

if only the first term in (5) is used. Dividing out the right-hand side yields a series of terms in z^{-i} from which proper spacings can be determined. It should be noted, however, that an exact solution to (12) is not always possible. A general discussion of this problem and of the types of excitation functions h(s) that can be approximated in this manner is included in the next section.

Use of (5) can be justified only over the portion of the visible range given by $-\pi/2 < \psi < +\pi/2$. In general this range includes the main beam and inner sidelobes of the array pattern. Unless modified these procedures are not applicable outside of the restricted range. However, it has been shown in several examples that when the objective is to achieve a narrow beam and low sidelobes the outer sidelobe structure is not adversely affected by the use of (5) and the small-angle approximations in the analytical method.[3]

An example of the correspondence indicated in (12) is illustrated in Fig. 1. Figure 1a is a plot of the array factor given in (10) with n = 16 and A = 0.2 corresponding to an equally excited unequally spaced array. Figure 1b is a plot of the array factor corresponding to the

equivalent equally spaced unequally excited array. The amplitude distribution was determined by solving (12) for $h(s)$ with $g(s)$ as given in (9) with $n = 16$. The close agreement between the two patterns over the nominal visible range of $0 \leq |\psi| \leq \pi$ is noted.

III. DISCUSSION

For an exact solution to (12) with $g(0) = 0$ it is necessary that the $(n-1)$ spacing parameters $g(i)$ satisfy

$$
\begin{aligned}
h(0) &= 1 - \frac{2A}{\pi} g(1) \\
h(1) &= 1 - \frac{2A}{\pi} g(2) \\
&\vdots \\
h(i) &= 1 + \frac{2A}{\pi} [g(i-1) - g(i+1)] \quad \text{for } i = 2, 3, \ldots, (n-2) \\
&\vdots \\
h(n-1) &= 1 + \frac{2A}{\pi} [g(n-2)] \\
0 &= g(n-1)
\end{aligned}
\tag{13}
$$

The first $(n-1)$ of these can be solved yielding

$$
\begin{aligned}
\frac{2A}{\pi} g(i) &= \frac{i+1}{2} - h(0) - h(2) - \ldots - h(i-1) \quad (i \text{ odd}) \\
\frac{2A}{\pi} g(i) &= \frac{i}{2} - h(1) - h(3) - \ldots - h(i-1) \quad (i \text{ even})
\end{aligned}
\tag{14}
$$

The remaining two of (13) are for n even

$$
\begin{aligned}
h(n-1) &= 1 + \frac{n-2}{2} - h(1) - h(3) - \ldots - h(n-3) \\
\frac{2A}{\pi} g(n-1) &= 0 = \frac{n}{2} - h(0) - h(2) - \ldots - h(n-2)
\end{aligned}
\tag{15}
$$

and for n odd

$$h(n-1) = 1 + \frac{n-1}{2} - h(0) - h(2) - \dots - h(n-3)$$
$$\frac{2A}{\pi} g(n-1) = 0 = \frac{n-1}{2} - h(1) - h(3) - \dots - h(n-2) \qquad (16)$$

Rearranging terms, for n even (15) becomes

$$h(1) + h(3) + \dots + h(n-1) = \frac{n}{2}$$
$$h(0) + h(2) + \dots + h(n-2) = \frac{n}{2} \qquad (17)$$

and, for n odd (16) reduces to

$$h(0) + h(2) + \dots + h(n-1) = \frac{n+1}{2}$$
$$h(1) + h(3) + \dots + h(n-2) = \frac{n-1}{2} \qquad (18)$$

Then, since the normalization condition

$$\sum_{i=0}^{n-1} h(i) = n \qquad (19)$$

must be satisfied it is evident that a solution to (12) is possible if either of the equations in (17) or (18) is satisfied.

If the amplitude distribution h(s) is symmetrical about the center of the array then for n even

$$\begin{aligned} h(0) &= h(n-1) \\ h(1) &= h(n-2) \\ &\vdots \\ h(\tfrac{n}{2} - 1) &= h(\tfrac{n}{2}) \end{aligned} \qquad (20)$$

and for n odd

$$\begin{aligned} h(0) &= h(n-1) \\ h(1) &= h(n-2) \\ &\vdots \\ h(\tfrac{n-3}{2}) &= h(\tfrac{n+1}{2}) \end{aligned} \qquad (21)$$

Inserting (20) and (21) into the normalization condition (19) yields

$$\sum_{i=0}^{n-1} h(i) = 2[h(0) + h(2) + \ldots + h(n-2)] = n \qquad (22)$$

for n even, and

$$\sum_{i=0}^{n-1} h(i) = 2[h(0) + h(1) + \ldots + h(\tfrac{n-3}{2})] + h(\tfrac{n-1}{2}) = n \qquad (23)$$

for n odd. Thus, it can be said that for n even an exact solution to (12) is possible if the equivalent amplitude distribution h(s) is symmetrical about the array center. For n odd an exact solution is possible only if either of the relations in (18) is satisfied in addition to the normalization condition.

IV. CONCLUSION

It has been shown that under certain conditions the Z-transform can be used to simplify the analytical treatment of unequally spaced antenna arrays. The method applies to both analysis and synthesis problems and allows many of the results originally derived for equally spaced arrays to be applied.

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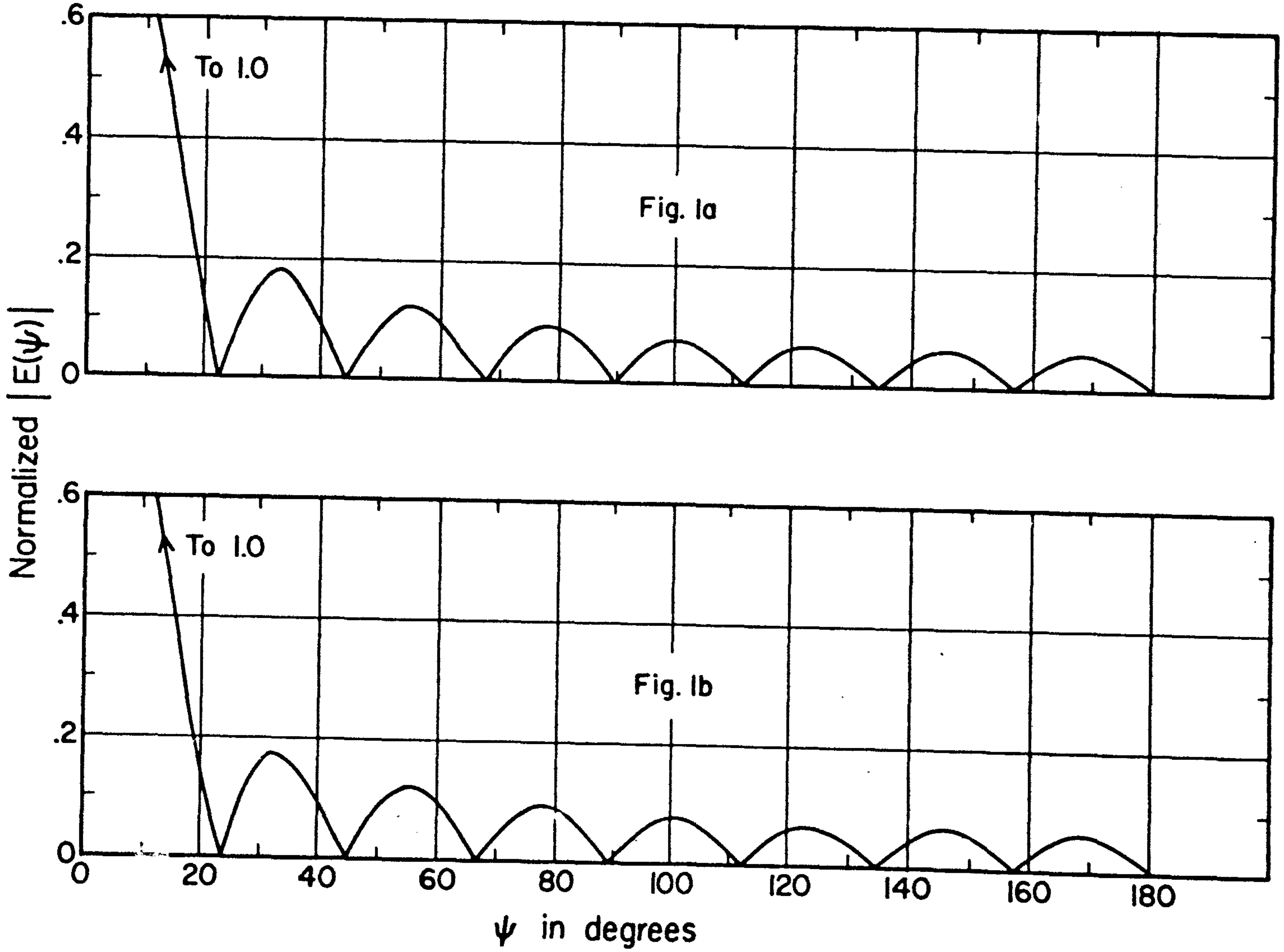

Fig. 1a - The Array Factor Given in (10) With $n = 16$, $A = 0.2$.

Fig. 1b - The Array Factor of the Equivalent Unequally Excited Array.

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